# Algorithms and Complexity Results for the 0–1 Knapsack Problem with Group Fairness

Alberto Santini[1] [0000−0002−0440−0357], Enrico Malaguti[2], and Paolo Paronuzzi[2]

[1] Universitat Pompeu Fabra, 08005 Barcelona, Spain alberto.santini@upf.edu
[2] University of Bologna, 40136 Bologna, Italy
{enrico.malaguti,paolo.paronuzzi}@unibo.it

**Abstract. Introduction.** This paper presents an exact algorithm for a generalisation of the classical 0–1 Knapsack Problem (KP) in which items are partitioned into classes, and fairness constraints affect the number of items that can or must be chosen from each class. In the KP, one is given $n$ items, each one having a profit $p_j \in \mathbb{N}$ and weight $w_j \in \mathbb{N}$, to be packed into a knapsack of capacity $c \in \mathbb{N}$ while maximising the packed profit. Patel et al. [1] introduced a family of problems, collectively called KP with Group Fairness (KPGF), in which the items are partitioned into $\ell$ classes $J_1, \dots, J_\ell$. The main idea behind the KPGF is that the items packed in the knapsack should represent each class fairly. To this end, the authors consider three different problems in which each class has associated upper and lower bounds on the total weight, total profit or number of items chosen from the class. In this paper, we generalise this idea further by associating a resource consumption value $h_j \in \mathbb{N}$ to each item and imposing lower and upper bounds on the total resource consumption of packed items of each class. The three problems introduced by Patel et al. [1] then correspond to the special cases in which, for each item $j$, $h_j = w_j$, $h_j = p_j$, and $h_j = 1$, respectively.

**Main application.** The KPGF can be used to determine a fair allocation of economic resources. Here, we highlight one use case which is particularly important due to its timeliness and social impact: participatory budgeting (PB). A participatory budget is a decision-making process in which ordinary citizens decide how to allocate part of a local government's budget. Contemporary PB was pioneered in the '90s by the Brazilian city of Porto Alegre and is now used in thousand of major and small cities and communities around the world, such as New York City, Barcelona, Bologna, Stuttgart and Zaragoza. In a typical PB process at the municipal level, citizens propose interventions they would like to see implemented in their neighbourhood, such as a new bike lane or renovating a public space. Municipality technicians then filter the technically feasible interventions and assign an estimated budget to each one. Then, a public voting phase starts, during which citizens can vote on one or multiple projects they would like to be implemented. At the end of this phase, the municipality selects which interventions to perform, limited by the available budget and considering citizens' preferences. The problem of selecting the subset of interventions to implement can be modelled as a KP in which the items are the interventions, the capacity is the available budget, the weights are the intervention costs, and the

profits are the number of votes gathered. Still, such a system could result in one or a few neighbourhoods receiving most interventions while others do not. Therefore, the municipality is usually interested in establishing rules that ensure that the implemented projects affect all parts of the city. In this case, the problem becomes a KPGF in which the item classes are the neighbourhoods, and many different fairness principles can be used. For example, fairness could be quantified by the number of projects (resulting in a KPGF with all $h_j = 1$), the budget (resulting in a KPGF with $h_j = w_j$), or some other criterion, such as the number of positively affected citizens (resulting in a KPGF in which the values $h_j$ do not coincide with other problem data).

**Methodology and contributions.** In our paper [2], we generalise the KPGF introduced by Patel et al. [1] and present models and an exact solution algorithm for this problem. Moreover, we settle the open question on the complexity of the KPGF by proving that it is weakly $\mathcal{NP}$-hard. First, we introduce a compact and an extended formulation based on Dantzig-Wolfe reformulation. The extended formulation has a number of variables which is exponential in $n$. However, we show that only a pseudopolynomial number of them are required to solve the KPGF to optimality. We exploit this property to devise a solution algorithm outperforming the direct application of a state-of-the-art mixed-integer programming solver to the compact formulation, as shown by the results.

**Keywords:** packing · knapsack problems · fairness · integer linear programming · dynamic programming.

## References

1. Deval Patel, Arindam Khan, and Anand Louis. Group fairness for knapsack problems. In: Frank Dignum, Alessio Lomuscio, Ulle Endriss, and Ann Nowé, editors, Proceedings of the 20th International Conference on Autonomous Agents and Multi-Agent Systems, pages 1001–1009. International Foundation for Autonomous Agents and Multiagent Systems (2021).
2. Enrico Malaguti, Paolo Paronuzzi, and Alberto Santini. Algorithms and complexity results for the 0–1 knapsack problem with group fairness. Optimization Letters (2026).